General conformal mapping and some torsion problems in comples plane

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Abstract— In this paper, we obtain the complex torsion functions for the cross section bounded by closed contour Γ in z- plane. The cross section is conformally mapped on the area inside the unit circle γ in ζ - plane by the rational mapping $Z = c\zeta \frac{1+\alpha\zeta^{m}+\delta\zeta^{2m}}{1+\beta\zeta^{n}}$ .Also, the torsional rigidity of the cylinder is determined. Many special cases are established and discussed from the work. The most of the author's works in this domain are considered as special case of this work.

Keywords and phrases: Complex torsion function- Complex plane- Conformal mapping- Torsional rigidity.

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1 INTRODUCTION

For many years contact and mixed problems, in the theory of elasticity, has been recognized as a rich and challenging subjected for study, see [1-4]. In addition, many different methods are established for solving the contact and mixed problems in elastic and thermo elastic problems. The books edited by Node et al.[5],Love [6], Popov [7], Aleksandrov et al.[8] and Parkus [9] contain many different methods to solve the contact problems, mixed problems and the problems of the complex plane in the theory of elasticity. These problems can establish from the initial value problems, the boundary value problems. More information for the elastic problems can be found in the work of Hetnarski et al. [10], Exadaktylos et al. [11, 12] and Abdou et al. [13, 14].

A very important problem in mathematical physics is the two-dimensional Dirichlet's problem for a simple closed curve. One of the situation from which it arisen is the classical Saint-Venant torsion problem of a homogeneous right cylinder of length *l* with lateral surface free of external load, when a twisting motion is applied to the bases of the beam which is subected to the stress. The calculation of these stresses is called the torsion problem. The stresses can cause deformations and potential failure of the beam and therefore the solution to the torsion problem is important in practice.

There are two common approaches to doing these calculations: numerical techniques that calculate approximate solutions and analytic techniques that lead to exact solution.

Analytic solutions, when it can be determined, are often preferable because it exact and can usually be calculatedquickly.

Several authors used various methods, to obtain the solution of the torsion problems in exact and closed forms. Some of authors used Laurent's theorem to express the solution as a power series, see Parkus [9] and Exadaktylos et al. [11, 12]. Others used complex variables method to obtain the solution of torsion problems in the form of two complex functions, see Muskhelishvili [15] and Abdou et al. [16-18].

In this paper, complex variable method is used to obtain

closed, exact expressions for a homogeneous isotropic cylindrical bar using general conformal mapping. The stress functions,

and the torsional rigidities and numerical values are calculated in some cases. Several figures are sketched showing the shapes of cross sections corresponding to special values of the parameters involved.

2 BASIC EQUATIONS:

Consider a homogeneous isotropic cylindrical bar, subject to no body forces and whose cross section in the xy- plane is bounded by a simple closed curve Γ, the generators of the bar are then parallel to z- axis.

It is known that, see Muskhelishvili [15], the complex torsion function, is given by

$$\Omega(z) = \Phi(x,y) + i\,\Psi(x,y); \quad z = x+iy \tag{1}$$

Where, the harmonic function $\Psi(x,y)$ satisfies the boundary conditions

$$\Psi(x,y) = \frac{1}{2} Z(\zeta)\bar{Z}(\zeta), + a\,constant\ on\,\Gamma; \quad \zeta = \rho e^{i\psi} \tag{2}$$

In terms of the variable ζ , the torsion function $\Omega(z)$ takes the form

$$\Omega(z) = W(\zeta) = \frac{1}{2\pi}\int_{\gamma} \frac{Z(\sigma)\bar{Z}(\sigma^{-1})}{\sigma-\zeta} d\sigma + a\,(Cons.) \tag{3}$$

Where $W(\zeta)$ is analytic in the interior of the circle $|\zeta| = 1$.

he twisting couple N is given by

$$N = \tau D, \quad D = \mu(I+J) \tag{4}$$

Where, τ is the constant twist per unit length, μ is the rigidity of the material of the bar, and D is the torsional rigidity of the cylinder. In addition, I is the polar moment of inertia for the cross section, where

$$I = \int_{S} z\bar{z}\,dS = -\frac{i}{4}\int_{\gamma} Z(\sigma)Z'(\sigma)\bar{Z}^{-2}(\sigma^{-1})d\sigma; \; J = \frac{i}{2}\int_{\gamma} Z(\sigma)\bar{Z}(\sigma^{-1})W'(\sigma)d\sigma \tag{5}$$

3 ELASTIC BAR WITH A UNIFORM CROSS SECTION

Consider the cross section bounded by closed contour Γ in z- plane, which is conformally mapped, on the area inside the unit circle γ in ζ - plane by the rational mapping

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$$Z = c\zeta \frac{1+\alpha\zeta^m + \delta\zeta^{2m}}{1+\beta\zeta^n}, \quad (c>0) \qquad (6)$$

Here, m and n are positive integers; while α, δ *and* β are real parameters restricted such that $Z'(\zeta)$ does not vanish or become infinite inside γ. The parametric equations of Γ are

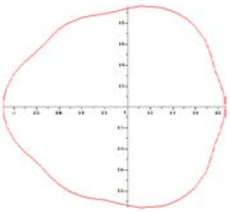
$$\frac{x}{c} = \frac{F(\psi)}{H(\psi)}; \quad \frac{y}{c} = \frac{G(\psi)}{H(\psi)}; \qquad (7)$$

Where, $H(\psi) = 1+\beta^2 + 2\beta\cos n\psi$

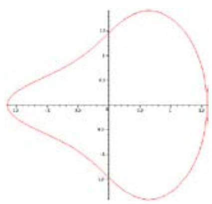
and

$$F(\psi) = \cos\psi + \beta\cos(n-1)\psi + \alpha\cos(m+1)\psi + \delta\cos(2m+1)\psi + \alpha\beta\cos(m-n+1)\psi + \delta\beta\cos(2m-n+1)\psi;$$

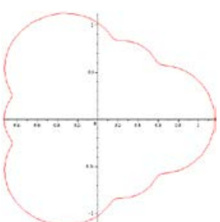
$$G(\psi) = \sin\psi - \beta\sin(n-1)\psi + \alpha\sin(m+1)\psi + \delta\sin(2m+1)\psi + \alpha\beta\sin(m-n+1)\psi + \delta\beta\sin(2m-n+1)\psi$$



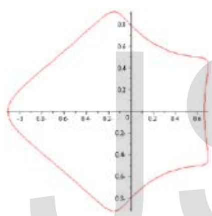
$(Fig.1: \alpha = .15, \delta = -.03, \beta = .08, m = 3, n = 1, c = 1);$



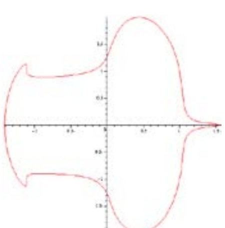
$(Fig.2: \alpha = 1/7, \delta = 1/8, \beta = 1/15, m = 1, n = 5, c = 1);$



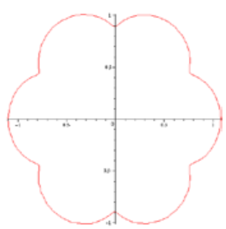
$(Fig.3: \alpha = -1/6, \delta = 1/5, \beta = 1/13, m = 3, n = 5, c = 1)$



$(Fig.4: \alpha = 0.2, \delta = -.0.2, \beta = .0.1, m = 1, n = 3, c = 1)$



$(Fig.5: \alpha = 0.09, \delta = -.0.02, \beta = 0.01, m = 2, n = 1, c = 1)$



$(Fig.6: \alpha = 0.19, \delta = -.0.1, \beta = .0.1, m = 1, n = 5, c = 1)$

Figs. (1-6) contain some shapes for the parametric equations (7) for different values α, δ, β, m *and* n.

To obtain the complex torsional function for certain curvilinear cross section for the general conformal mapping (6), we must discuss thefollowing three cases:

3.1 **Case (I):** $(n > 2m)$

In this case, the complex torsion function is assumed in the form

$$W(\zeta) = \frac{c^2 i}{1+\beta\zeta^n}[A_0 + A_1\zeta^m + A_2\zeta^n + A_3\zeta^{n-m} + A_4\zeta^{2m} + A_5\zeta^{n-2m}] \qquad (8)$$

Where $A_i, i = 0,1,2,3,4,5$ are real constant to be determined from (2) on γ. After determining A_i's the formula (8) yields

$$W(\zeta) = \frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}[1+\alpha^2+\delta^2+2\alpha(1+\delta)\zeta^m - \beta(1+\alpha^2+\delta^2)\zeta^n - 2\alpha\beta(1+\delta)\zeta^{n-m} + 2\delta\zeta^{2m} - 2\delta\beta\zeta^{n-2m}] \qquad (9)$$

Many special cases can be derived from case (a). For example

●**(I-1):**Let, in (6), m=2, n=5, we have

$$Z = c\zeta\frac{1+\alpha\zeta^2+\delta\zeta^4}{1+\beta\zeta^5}. \qquad (10)$$

The corresponding complex torsion function is

$$W(\zeta) = \frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^5)}[1+\alpha^2+\delta^2 - 2\delta\beta\zeta + 2\alpha(1+\delta)\zeta^2 - 2\alpha\beta(1+\delta)\zeta^3 + 2\delta\zeta^4 - \beta(1+\alpha^2+\delta^2)\zeta^5] \qquad (11)$$

●**(I-2):**Let in the conformal mapping (3.1) $\alpha = 0$, to obtain the conformal mapping

$$Z = c\zeta\frac{1+\delta\zeta^{2m}}{1+\beta\zeta^n}, \quad (c>0) \qquad (12)$$

The corresponding complex torsion function takes the following form:

$$W(\zeta) = \frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}[1+\delta^2 - \beta(1+\delta^2)\zeta^n + 2\delta\zeta^{2m} - 2\delta\beta\zeta^{n-2m}].$$

●**(I-3):** Let in (6), $\delta = 0$, to get the mapping

$$Z = c\zeta\frac{1+\alpha\zeta^m}{1+\beta\zeta^n}, \quad (c>0) \qquad (13)$$

The corresponding complex function is

$$W(\zeta) = \frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}[1+\alpha^2 + 2\alpha\zeta^m - \beta(1+\alpha^2)\zeta^n - 2\alpha\beta\zeta^{n-m}]$$

●**(I-4):** Let in (7), $\beta = 0$, to obtain

$$Z = c\zeta(1+\alpha\zeta^m + \delta\zeta^{2m}), \quad (c>0). \qquad (14)$$

The complex torsion function takes the form:

$$W(\zeta) = \frac{c^2 i}{2}[1+\alpha^2+\delta^2+2\alpha(1+\delta)\zeta^m + 2\delta\zeta^{2m}]$$

Many special and different cases can be established and discussed from the conforming mapping (12), (13) and (14) for the different values of α, δ, β, m *and* n.

There are however two cases of particular interest.

3.2 Case (II): $(n < m)$;

In this case, the complex torsion function is assumed in the form:

$$W(\zeta) = \frac{c^2 i}{1+\beta\zeta^n}\sum_{s=0}^{2m} A_s\zeta^s \qquad (A_s\text{'s are real constants}) \quad (15)$$

Using the boundary condition (2), then equating the coefficients of the term $\cos\mu\psi;\ \mu = 0,1,...,2m$, on both sides, we have (2m+1) equations are sufficient to determine the A_s's. For example: for the conformal mapping

$$Z = c\zeta\frac{1+\alpha\zeta^3+\delta\zeta^6}{1+\beta\zeta^2}, \quad (c>0) \qquad (16)$$

The corresponding complex torsion function is
:

$$W(\zeta)=\frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^2)}\{1+\alpha^2+\delta^2-2\beta^2\delta-2\alpha\beta(1+\delta)(1+\beta)\zeta$$
$$+\beta(1+\alpha^2+\delta^2-2\delta\beta)\zeta^2+2(1-\beta^2)[\alpha+\alpha^2-\delta\beta\zeta+\delta\zeta^3]\zeta^3\}$$

There are however two cases of particular interest.

●**(II-1):** When $m>n$ with a common integer factor i.e., $m=pn;\ p=1,2,3,...,N$ *i.e.*

$$Z=c\zeta\frac{1+\alpha\zeta^{pn}+\delta\zeta^{2pn}}{1+\beta\zeta^n},\quad (p=2,3,...,N;n=1,...,M;c>0). \quad (17)$$

The corresponding complex torsion function is

$$W(\zeta)=\frac{c^2 i}{(1+\beta\zeta^n)}[\sum_{s=0}^{2p}A_s\zeta^{sn}];\quad A_{2p}=\delta; \quad (18)$$

where $A_j, j=0,1,..2p-1$ are unknown constants can be determined using Eqs.(2)-(3)can be obtained in the form

$$A_0=\frac{1}{2(1-\beta^2)}\{1+\alpha^2+\delta^2+2(-\beta)^p[\alpha+\alpha\delta+(-\beta)^p\delta]\},$$

$$A_1=\frac{1}{2(1-\beta^2)}\{2(-\beta)^{p-1}[\alpha+\alpha\delta+(-\beta)^p\delta]-\beta(1+\alpha^2+\delta^2)\}$$

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$$A_{p-\nu}=(-\beta)^\nu[\alpha+\alpha\delta+(-\beta)^p\delta];\quad A_{2p-\nu}=(-\beta)^\nu\delta.$$

Where, the final form of the corresponding complex torsion function takes the form

$$W(\zeta)=\frac{c^2 i}{(1+\beta\zeta^n)}[A_0+A_1+A_{p-\nu}+A_{2p-\nu}]; \quad (19)$$
$$(\nu=1,2,...,p-1;p-\nu\neq 1).$$

Many special cases can be derived from this case

●**(II-1-a)**: Let in (17), $\delta=0$, we have

$$Z=c\zeta\frac{1+\alpha\zeta^{pn}}{1+\beta\zeta^n},\quad (c>0;\ p=2,...,N)$$

The corresponding complex torsion function yields

$$W(\zeta)=\frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}\{1+\delta^2+2(-\beta)^{2p}\delta$$
$$-\beta[1+\delta^2+2(-1)^{2p}\beta^{2p-2}\delta]\zeta^2+2\delta(1-\beta^2)\sum_{\nu=0}^{2p-2}(-\beta)^\nu\zeta^{(2p-\nu)n}\}$$

●**(II-1-b):** Let in (17), $p=1$, we have

$$Z=c\zeta\frac{1+\alpha\zeta^n+\delta\zeta^{2n}}{1+\beta\zeta^n};n=0,1,2,...,N$$

The corresponding complex torsion function becomes

$$W(\zeta)=\frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}\{1+\alpha^2+\delta^2-2\beta(\alpha+\alpha\gamma-\beta\delta)$$
$$+[2\alpha(1+\delta)-\beta(1+\alpha^2+\delta^2+2\delta)]\zeta^n+2(1-\beta^2)\delta\zeta^{2n}\}$$

3.3 Case (II.2): ($m>n$)

Let without common factor $m>n$ i. e., we always find an integer p such that $(p+1)n>m>pn$.

In such case, we assume the torsion function in the form

$$W(\zeta)=\frac{c^2 i}{(1+\beta\zeta^n)}[A_0+A_1\zeta^n+A_2\zeta^{(p+1)n-m}$$
$$+A_3\zeta^{(2p+1)n-2m}+\sum_{s=0}^{p}B_s\zeta^{m-sn}+\sum_{s=0}^{2p}B_s\zeta^{2m-sn}]$$

The constants $A_0,A_1,A_2,A_3,B_0,B_1,...,B_p,C_0,C_1,...,C_{2p}$ are real and can be determined from the condition (2). After obtaining the values of the constant, the torsion function is

$$W(\zeta)=\frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}\{(1+\alpha^2+\delta^2)(1-\beta\zeta^n)+2(-\beta)^p\alpha(1+\delta)$$
$$X[\zeta^{m-np}-\beta\zeta^{(p+1)n-m}]+2\alpha(1-\beta^2)\sum_{\nu=0}^{p-1}(-\beta)^\nu\zeta^{m-n\nu}$$
$$+2(-\beta)^{2p}\delta[\zeta^{2m-np}-\beta\zeta^{(2p+1)n-2m}]+2\delta(1-\beta^2)\sum_{\nu=0}^{2p-1}(-\beta)^\nu\zeta^{2m-n\nu}\} \quad (20)$$

Many different special cases can be derived from (20), when $(\alpha=0;\delta\neq 0,\beta\neq 0)$ $(\alpha\neq 0,\delta=0,\beta\neq 0);(\alpha\neq 0,\delta\neq 0,\beta=0)$, and for different values of m and n.

3.4 Case(III): $(2m>n>m;\ n=3,...,N;m=2,3,...,N-1)$

In this case, the complex torsion function can be assumed in the form

$$W(\zeta)=\frac{c^2 i}{(1+\beta\zeta^n)}[A_0+A_1\zeta^n+A_2\zeta^{2n-2m}+A_3\zeta^m+A_4\zeta^{n-m}+A_5\zeta^{2m}+A_6\zeta^{2m-n}]$$

After determining the constants A_0 to A_6 and inserting the results in the above, we have

$$W(\zeta)=\frac{c^2 i}{2(1-\beta^2)(1+\beta\zeta^n)}\{(1+\alpha^2+\delta^2)(1-\beta\zeta^n) \quad (21)$$
$$+2\delta[(1-\beta^2)\zeta^{2m}+\beta^2\zeta^{2n-2m}-\beta\zeta^{2m-n}]+2\alpha(1+\delta)(\zeta^m-\beta\zeta^{n-m}\}$$

4. TORSION RIGIDITY FOR SOME CROSS SECTION

In this section, we will determine the torsion rigidity D for cross sections that can be mapped on the unit circle γ by the rational mapping function

$$Z=c\zeta\frac{1+\alpha\zeta^{3n}}{1+\beta\zeta^n};\quad c>0 \quad (22)$$

With its corresponding complex torsion function

$$W(\zeta)=\frac{ic^2}{2(1-\beta^2)(1+\beta\zeta^n)}\{1+\alpha^2-2\alpha\beta^3 \quad (23)$$
$$+\beta(2\alpha\beta-\alpha^2-1)\zeta^n+2\alpha(1-\beta^2)\zeta^n(\zeta^n-\beta)\}$$

Using the first formula of (5), we obtain

$$I = \frac{-ic^4}{4}\int_\gamma G(\sigma)d\sigma;$$

$$G(\sigma) = \frac{(1+\alpha\sigma^{3n})(\alpha+\sigma^{3n})^2}{\sigma^{4n+1}(1+\beta\sigma^n)^3(\beta+\sigma^n)^2}[1-\beta(n-1)\sigma^n + \alpha(3n+1)\sigma^{3n} + \alpha\beta(2n+1)\sigma^{4n}] \quad (23)$$

The integrand (23) has a pole of order (4n+1) at the origin, and n double poles at λ_s ,s=1,2,…n where $\lambda_s^n = -\beta$; $(|\beta| < 1)$.

Let P_s denotes the residue of the integrand at the points λ_s and P denotes the residue at the origin, then we have

$$I = \frac{\pi c^4}{2}[P + \sum_{s=1}^{n} P_s] \quad (24)$$

In order to get the residue P we expand the integrand $G(\sigma)$ in power of σ . Hence the coefficient of σ^{-1} in the resulting expression is

$$P = \frac{1}{\beta^6}\{5\alpha^2 + 4\alpha^2(n+1)\beta^2 - 2(2\alpha^2 + 3n\alpha^2 + 2)\alpha\beta^3 + 9\alpha^2(n+1)\beta^4 - 2\alpha(2+2\alpha^2+n+4\alpha^2 n)\beta^3 + 9\alpha^2(n+1)\beta^4 - 2\alpha(2+2\alpha^2+n+4\alpha^2 n)\beta^5 + 4\alpha^2(2+3n)\beta^6 + 5\alpha^2(1+2n)\beta^8\}. \quad (25)$$

For the residues, P_s we set $\sigma = \zeta_s + t$ in the integrand (23) and expand the resulting in powers of t. After some algebraic work the coefficient of t^{-1} is found in the form

$$P_s = \frac{(\alpha-\beta^3)}{n\beta^6(1-\beta^2)^4}\{(1-\beta^2)[-1-(n-1)\beta^2 + (3n+1)\alpha\beta^3 + (2n+1)\alpha\beta^5] \,X\, (\alpha + 2\alpha^2\beta^3 + 5\beta^3 - 8\alpha\beta^6) + (\alpha-\beta^3)(1-\alpha\beta^3) \,X\, [-4 + (10-3n)^2\beta + 6(n-1)\beta^4 + \alpha\beta^3(3n+1)(1-4\beta^2) + 3(2n+1)\alpha\beta^7]\} \quad (26)$$

Inserting (25), (26) in (24), we have the polar moment of inertia in the final form

$$I = \frac{\pi c^4}{2(1-\beta^2)^4}\{1 + 2\alpha^2(2+3n) + \alpha^4(1+3n) + 2\beta^2 n(1+4\alpha^2+\alpha^4) - 8\alpha\beta^3[2+3n+2\alpha^2(1+3n)] - \beta^4[(1-n)+2\alpha^2(2+n)+\alpha^4(1+2n)] + 4\alpha\beta^5[\alpha^2(6+13n)+2(3+2n)] + 14\alpha^2\beta^6(1+3n) - 4\alpha\beta^7[2\alpha^2(2+n)(1+2n)] - 8\alpha^2\beta^8(3+7n) + 10\alpha^2\beta^{10}(1+2n)\}. \quad (27)$$

Introducing (4.1), (4.2) in the second formula of (2.5), we get

$$J = \frac{nc^4 i}{2(1-\beta^2)}\int_\nu h(\sigma)\{\beta(-1-\alpha^2+\alpha\beta+\alpha\beta^3) - \alpha(1-\beta^2)[2\beta\sigma^n - (3-\beta^2)\sigma^{2n} - 2\beta\sigma^{3n}]\}d\sigma;$$

$$h(\sigma) = \frac{(1+\alpha\sigma^{3n})(\alpha+\sigma^{3n})}{\sigma^{n+1}(1+\beta\sigma^n)(\beta+\sigma^n)} \quad (28)$$

Inside ν , the integrand has simple poles at $\lambda_s, s = 1,2,...,n;\ \ \lambda_s^n = -\beta,\ \ |\beta < 1|$ and pole of order (n+1) at the origin.

Let Q and Q_s denote the residues at the origin and λ_s , respectively.

Then

$$J = -2c^4\pi(Q + \sum_{s=1}^{n} Q_s) \quad (29)$$

Following the previous way of determining (27), (28), we have

$$Q = \frac{\alpha}{\beta}\{1 + \alpha^2 - 3\alpha\beta - 2\alpha\beta^3 + 3\beta^2(1+\alpha^2) - 3\alpha\beta^5\} \quad (30)$$

And

$$Q_s = \frac{(1-\alpha\beta^3)(\alpha-\beta^3)}{n\beta(1-\beta^2)^3}\{-1-\alpha^2+6\alpha\beta-7\alpha\beta^3+3\alpha\beta^5\} \quad (31)$$

Introducing (30), (31) in (29), we get

$$J = -\frac{\pi nc^4}{(1-\beta^2)^4}\{3\alpha^2 + (1+\alpha^2)^2\beta^2 - 12(1+\alpha^2)\beta^3 - 3\alpha^2\beta^4 + 14(1+\alpha^2)\beta^5 + 12\alpha^2\beta^6 + 6\alpha(1+\alpha^2)\beta^7 - 14\alpha^2\beta^8 + 6\alpha^2\beta^{10}\}. \quad (32)$$

The torsion rigidity D, is obtained in the final form (4.12)

$$D = \frac{\pi\mu c^4}{(1-\beta^2)^4}\{1 + 4\alpha^2 + \alpha^4(1+3n) + [n-1-(2n+1)\alpha^4]\beta^4 + 4n\alpha^2\beta^2 - 8\alpha^2\beta^3(2+3n) - 2\alpha\beta^3(8-2\alpha\beta-6\beta^2) + 12\alpha\beta^5[2\alpha^2(n+1)-(n-2)] + 2\alpha^2\beta^6(7+9n) - 4\alpha\beta^7[\alpha^2(n+2)+2(1-n)] - 4\alpha^2\beta^8(6+7n) + 2\alpha^2\beta^9(5+4n)\} \quad (33)$$

Which is a parabolic function of n.

As an important special case, for the epitrochoid cross section, we let $\beta = 0$ in (33) to get

$$D = \frac{\pi\mu c^4}{2}\{1 + 4\alpha^2 + (3n+1)\alpha^4\} \quad (34)$$

In addition, let $\alpha = \beta^3$, in (22), we get the following conformal mapping

$$Z(\zeta) = c\zeta(1 - \beta\zeta^n + \beta^2\zeta^{2n}) \quad (35)$$

The corresponding torsion rigidity to (35) is

$$D = \frac{\pi\mu c^4}{2}\{1 + 4\beta^2 + (n+9)\beta^4 + 4(n+1)\beta^6 + (2n+1)\beta^8\} \quad (36)$$

5. SHEARING STRESSES FOR SOME CROSS SECTIONS:

The Shearing stresses $\hat{\rho}Z$ and $\hat{\psi}Z$ at any point of the cross section are given by, see [14]

$$\hat{\rho}Z - i\hat{\psi}Z = -\frac{\tau\mu\zeta}{|Z'(\zeta)|}[W'(\zeta) - iZ'(\zeta)\bar{Z}(\bar{\zeta})] \quad (37)$$

Using the conformal mapping (22) and (23) in (37), we have

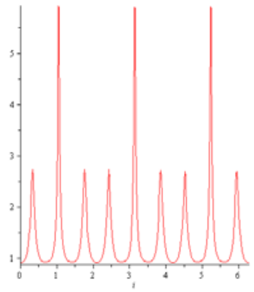
$$\hat{\rho}Z - i\hat{\psi}Z = \frac{c\mu\tau i}{T}\{\frac{n}{\rho(1-\beta^2)}[(-1-\alpha^2+\alpha\beta+\alpha\beta^3)\beta\zeta^{-n} - \alpha(1-\beta^2) \,X\, (2\beta - (3-\beta^2)\zeta^n - 2\beta\zeta^{2n})](\zeta^n + \rho^{2n}\beta)^2 - \rho[1 + (1-n)\beta\zeta^n + (3n+1)\alpha\zeta^{3n} + (2n+1)\alpha\beta\zeta^{4n}](1+\alpha\bar{\zeta}^{3n})(1+\beta\bar{\zeta}^n)\}. \quad (38)$$

$$T = |1 + (1-n)\beta\zeta^n + (3n+1)\alpha\zeta^{3n} + (2n+1)\alpha\beta\zeta^{4n}||1+\beta\zeta^n|^2$$

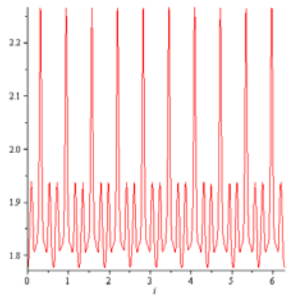
Putting $\rho = 1$, in (38), then after considerable amount of algebraic work, we have:

$$(\hat{\rho}Z)_{\rho=1} = 0;\quad (\hat{\psi}Z)_{\rho=1} = \frac{c\mu\tau}{(1-\beta^2)}\frac{\sum_{s=0}^{3} A_s \cos sn\psi}{[\sum_{s=0}^{4} B_s \cos sn\psi]^{1/2}}; \quad (39)$$

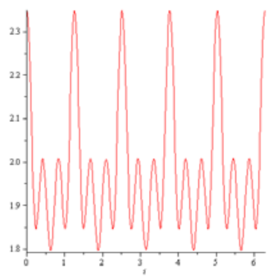
$$A_0 = 1 - 2n\alpha\beta^3 + (3n+1)\alpha^2 - (2n+1)\alpha^2\beta^2 - (1-n)\beta^2, A_1 = 2n\alpha\beta^2(1-\beta^2),$$
$$A_2 = -2n\alpha\beta(1-\beta^2), \quad A_3 = 2\alpha(1-\beta^2),$$
$$B_0 = 1 + (1-n)^2\beta^2 + (3n+1)^2\alpha^2 + (2n+1)\alpha^2\beta^2, \qquad (40)$$
$$B_1 = 2\beta[1-n+(3n+1)(2n+1)\alpha^2], \quad B_2 = 2\alpha\beta(1-n)(3n+1),$$
$$B_3 = 2\alpha[3n+1+(1-n)(2n+1)\beta^2], \quad B_4 = 2\alpha\beta(1+2n)$$



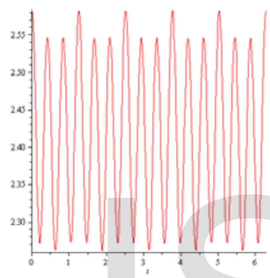
$(Fig.7: \alpha = 1/7, \beta = -1/15, \tau = 1/2, \mu = 3/7, n = 3, c = 1)$



$(Fig.8: \alpha = 1/7, \beta = 1/15, \tau = 1/2, \mu = 3/7, n = 10, c = 1)$



$(Fig.9: \alpha = 0.7, \beta = -0.1, \tau = 1/2, \mu = 3/7, n = 5, c = 1)$



$(Fig.10: \alpha = 0.9, \beta = -0.01, \tau = 1/2, \mu = 3/7, n = 15, c = 1)$

Figs. (7-10)

Figs(7-10) describe the shear stress $(\widehat{\psi Z})_{\rho=1}$ for different values of $\alpha, \beta, \tau, \mu,$ *and* n.

6 CONCLUSION:

From the above, we can deduce the following:

(**1**) In the theory of elasticity, in two-dimensional torsion problems, one of the most useful techniques is using the conformal mapping in the complex plane. The conformal mapping transforms the region into a simpler shape to get the analytical solutions without difficulties. Conformal mapping are widely used in plane linear elasticity because they help in transforming very complicated shapes into such simple one and allow the basic complex variable formation to extend to the transformation problem, thereby making the powerful methods of solutions developed for circular and half- plane regions to be applicable to these problems

(2) For the shearing stress $\widehat{\psi Z}$, the number of (n) leads to the same number of harmonic for fixed n, see Figs. (7-10**)**.

(3) For fixed n and different values of α and β, the function depends on the comparison between α and β. For example, if $\alpha \gg$ β, the top of harmonic are increasing.

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